

# Renormalized Free Energy on Space-time with Compact Hyperbolic Spatial Part

R. de Oliveira<sup>1</sup>

<sup>1</sup> Universidade Federal do Mato Grosso (UFMT)- MT-Brasil; rosevaldo.deoliveira@hotmail.com

## Abstract

In this paper we found the renormalized free energy of a interacting scalar field on a compact hyperbolic manifold explicitly. We have shown a complete expression of the free energy and entropy as a function of the curvature and the temperature. Carefully analyzing the free energy we have shown that there exist a minimum with respect to the curvature that depend on the temperature. The principle of minimum free energy give us an estimate of the connection between stationary curvature and temperature. As a result we obtain that the stationary curvature increases when the temperature increases too. If we start from an universe with very high curvature and temperature in the beginning, because of the principle of minimum free energy, the universe will reach a new situation of equilibrium for low temperature and low curvature. Consequently, the flat space-time is obtained for low temperature.

PACS number: 04.62.+v;11.10.Wx; 47.27.ef

Keywords: Quantum fields in curved spacetime, Finite-temperature field theory

## 1 Introduction

The theory of quantum fields in curved space-times have been considered by many authors[[1], [2], [3]], it deals with fields in a external gravitational field and can be considered as preliminary step toward the complete quantum theory of gravity.

The finite temperature effective potential in quantum field theory at finite temperature in curved space-time is the central object which should define the behavior of the early universe. Unfortunately it is difficult to calculate the finite temperature potential in a general curved space-time. Therefore it is very natural to deal with some specific spaces which are interesting from the cosmological viewpoint.

We have considered a ultrastatic space-time because the spatial section is a manifold with constant curvature, in such manifolds for a fixed value of the cosmological time, it describe locally spatially homogeneous isotropic universes [4]. The compact hyperbolic manifolds also have the constant curvature spatial section, but the spectrum of the relevant operator is, in general, not explicitly known<sup>1</sup>. However, there exist a mathematical tool, the Selberg trace formula [4], [6], [7] and [8], which permits to evaluate some interesting physically global quantities, like the vacuum energy or the one-loop finite and zero temperature effective potential.

Throughout the paper we will use the zeta-function technique for regularizing the path integral [4],[5],[21] and [22].

In this work we will consider a model of a self-interacting scalar field on a curved space-time with compact hyperbolic spatial part [13].

The finite temperature effects of this model are studied

by employing a complex integral representation for the finite temperature potential (or free energy) [13]-[19],[22].

Taking into account the principle of minimum free energy (or maximum of entropy) and analyzing the behavior of the renormalized free energy, we can deduce the connection between stationary curvature with the temperature.

## 2 Finite and zero temperature effective Potential

Before we restrict our attention to the hyperbolic manifolds  $\mathcal{M} = S^1 \times \mathbb{R}^1 \times H^2/\Gamma$ , let us define the zero and finite temperature effective potential. The Euclidean generating functional for Greens functions in a scalar field theory is given by, see ref. [1]

$$Z_E[J] = e^{-\frac{W_E[J]}{\hbar}} = N_E \int \mathcal{D}\varphi e^{-\frac{S_E[\varphi, J]}{\hbar}}, \quad (1)$$

where  $W_E[J]$  generates connected Greens functions. The Euclidean action is

$$S_E[\varphi, J] = \int d^4x \left[ \frac{1}{2} \partial_\mu \varphi \partial_\mu \varphi + \frac{1}{2} m^2 \varphi^2 + V(\varphi) - J\varphi \right]. \quad (2)$$

Let us define the following relation

$$\frac{\delta W_E[J]}{\delta J(x)} = -\phi_c(x), \quad (3)$$

<sup>1</sup> When the eigenvalues are known, the zeta function can be computed explicitly [see for example ref. [5]]

$\phi_c(x)$  is a classical variable.

The effective action is defined by

$$\Gamma_E[\phi_c] = W_E[J] + \int dx \phi_c(x) J(x). \quad (4)$$

In the one-loop approximation we find that the Euclidean generate functional is given by

$$Z_E[J] = e^{-\frac{S_E[\phi_c, J]}{\hbar}} [\det S_2^E \mu^2]^{-\frac{1}{2}}, \quad (5)$$

where

$$S_2^E = \{-\partial_\mu \partial_\mu + m^2 + V''(\phi_c)\} \delta(x_1 - x_2). \quad (6)$$

the constant  $\mu$  is introduced to become the logarithm dimensionless. The above expression can be proved to have only diagrams with one independent integral over the four momenta.

## 2.1 The zero temperature effective potential

We expect to be able to write  $\Gamma_E[\phi_c]$  in the form

$$\Gamma_E[\phi_c] = \int dx (-V_{eff}(\phi_c(x)) + \frac{1}{2} A(\phi_c(x)) \partial_\mu \phi_c(x) \partial_\mu \phi_c(x) + \dots). \quad (7)$$

If we set the source equal to zero, then the classical field takes a constant value  $\phi_c(x) = \phi_c^2$ . In this limit the effective action is given to be

$$\Gamma_E[\phi_c] = V_E V_{eff}(\phi_c), \quad (8)$$

where  $V_E$  is the Euclidian measure. The effective action is written as<sup>3</sup>

$$\begin{aligned} \Gamma_E[\phi] &= S_E[\phi] + \hbar \bar{\Gamma}_E^1[\phi_c] \\ V_{eff} &= V_{cl} + \frac{\hbar}{V_E} \bar{\Gamma}_E^1[\phi_c] \end{aligned} \quad (9)$$

Using the Eq.[11] we can write the effective potential as

$$V_{eff} = V_{cl} + \frac{\hbar}{2V_E} \ln [\det S_2^E \mu^2]. \quad (10)$$

<sup>2</sup> You can see looking at in the equation  $\frac{\delta \Gamma[\phi_c]}{\delta \phi_c} = 0$

<sup>3</sup> we need to remember that

$$\left[ \ln(Z_E[J]) - \ln \left( e^{-\frac{S_E[\phi_c, J]}{\hbar}} [\det A \mu^2]^{-\frac{1}{2}} \right) \right]_{J=0, \phi_c=cte} = 0$$

## 2.2 The finite temperature effective potential

The theory at finite temperature,  $T = \frac{1}{\beta}$ , can be obtained by compactifying the imaginary time  $\tau$  and assuming periodic conditions on the fields,  $\phi(\tau, x) = \phi(\tau + \beta, x)$ . The partition function of a scalar fields in thermal equilibrium at finite temperature  $T$ , (see ref.[2]) is defined as

$$Z_\beta = e^{-\frac{W_\beta[J]}{\hbar}} = N_E \int \mathcal{D}\varphi e^{-\frac{S_E[\varphi, J]}{\hbar}}, \quad (11)$$

where the field satisfy a periodic boundary condition  $\varphi(\tau + \beta) = \varphi(\tau)$ . We define the following relation

$$\phi_c(x) = -\frac{\delta W_\beta[J]}{\delta J(x)} = \frac{\hbar}{Z_\beta[J]} \frac{\delta Z_\beta[J]}{\delta J(x)} = \langle \hat{\phi}(x) \rangle_J \quad (12)$$

The finite temperature effective action is defined as

$$\Gamma_\beta[\phi_c] = W_\beta[J] + \int_0^\beta d\tau \int dx J(x) \phi_c(x) \quad (13)$$

By using of the result of the last section, to one loop approximation we can write the partition function as

$$Z_\beta = e^{-\frac{S_E[\phi_c, J]}{\hbar}} [\det S_{2,\beta}^E \mu^2]^{-\frac{1}{2}} \quad (14)$$

The free energy is given to be

$$F = -\frac{1}{\beta} \ln Z_\beta = \frac{S_E}{\hbar\beta} + \frac{1}{2\beta} [\det S_{2,\beta}^E \mu^2] \quad (15)$$

When  $\phi_c(x) = cte$  the free energy per unit volume is

$$\frac{\hbar F}{\Omega} = V_c + \frac{\hbar}{2\Omega\beta} \ln [\det S_{2,\beta}^E \mu^2] \quad (16)$$

where  $V_E = \beta\Omega$ . Thus we define the finite effective potential as

$$V_{eff}^\beta = \frac{\hbar F}{\Omega} = V_c + \frac{\hbar}{2\Omega\beta} \ln [\det S_{2,\beta}^E \mu^2] \quad (17)$$

$$V_{eff}^\beta = V_c + \frac{\hbar F_0}{\Omega} + \frac{\hbar F_\beta}{\Omega}.$$

$$V_c = \frac{\lambda\phi^4}{4!} + \frac{m^2\phi^2}{2} + \frac{\xi R\phi^2}{2}. \quad (18)$$

## 3 One-Loop Effective Potential on $\mathcal{M} = S^1 \times R^1 \times H^2/\Gamma$

With the purpose of find the one-loop effective potential, we will consider the representation of the zeta-function given by

$$\begin{aligned} \frac{\zeta(s|A)}{\Omega_1} &= \frac{\beta\Gamma(s-1)}{4\pi\Gamma(s)} \zeta(s-1|L_2) \\ &+ \frac{1}{4\pi\Gamma(s)\pi i} \int_{\Re z=C} \zeta_R(z) \Gamma\left(\frac{z-1}{2} + s - \frac{1}{2}\right) \\ &\times \Gamma\left(\frac{z}{2}\right) \left(\frac{\beta}{2}\right)^{-z+1} \zeta\left(\frac{z-1}{2} + s - \frac{1}{2}|L_2\right) dz. \end{aligned} \quad (19)$$

The one-loop effective potential is given by Eq(10), i.e

$$V_{eff} = V_c + V^{(1)}. \quad (20)$$

We know that the determinant of the operator is related to zeta-function as

$$\begin{aligned} V^{(1)} &= \frac{1}{2\Omega(\mathcal{M})} \ln \det(A\mu^2) \\ &= \frac{1}{2\Omega(\mathcal{M})} \lim_{s \rightarrow 0} \left[ -\frac{d\zeta(s|A)}{ds} + \zeta(s|A) \ln \mu^2 \right]. \end{aligned} \quad (21)$$

Now we can write the quantum correction by using the above results as

$$\begin{aligned} V^{(1)} &= V_0^{(1)} + V_\beta^{(1)} \\ &= \frac{1}{2\Omega(\mathcal{M})} \left\{ +\frac{\beta}{4\pi\Omega_1} [R_H^{-2} \mathcal{F}(0; \delta) \right. \\ &\quad \left. + \lim_{s \rightarrow 0} \frac{d}{ds} (R_H^{2(s-1)} \mathcal{F}(s; \delta)) \right] \\ &\quad - \frac{1}{4\pi^2 i \Omega_1} \int_{\Re z = C} \zeta_R(z) \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z-2}{2}\right) R_H^{z-2} \mathcal{F}\left(\frac{z}{2}; \delta\right) \\ &\quad \left. \left(\frac{\beta}{2}\right)^{-z+1} dz - \frac{\beta\Omega_1}{4\pi} R_H^{-2} \mathcal{F}(0; \delta) \ln \mu^2 \right\} \end{aligned} \quad (22)$$

The volume is  $\Omega(\mathcal{M}) = \beta\Omega_1 R_H^2 V(\mathcal{F})$ . In order to derive the zero temperature contribution we need to calculate the derivative of the Bytsenko function

$$\begin{aligned} \frac{d\mathcal{F}(s; \delta)}{ds} &= \frac{V(\mathcal{F})}{4\pi} \left[ \frac{\delta^{2(2-s)}(-2) \ln \delta}{(s-2)} + \frac{\delta^{2(2-s)}(-1)}{(s-2)^2} \right] \\ &\quad - \frac{V(\mathcal{F})}{2\pi} \frac{dL_1(\Delta^{1-s})}{ds} + \left\{ H(s) \left[ \frac{-\psi(s-1)}{\Gamma(s-1)\Gamma(2-s)} \right. \right. \\ &\quad \left. \left. + \frac{\psi(2-s)}{\Gamma(s-1)\Gamma(2-s)} \right] + \frac{H'(s)}{\Gamma(s)\Gamma(2-s)} \right\} \\ &\quad + T_0 \frac{dL_3(\Delta^{1-s})}{ds} \end{aligned} \quad (23)$$

Where we have defined

$$\begin{aligned} L_1(\Delta^{1-s}) &= \int_0^\infty r(1 - \tanh \pi r)(r^2 + \delta^2)^{1-s} dr \\ &= 2 \int_0^\infty \frac{r(r^2 + \delta^2)^{1-s}}{1 + e^{2\pi r}} dr, \end{aligned} \quad (25)$$

$$L_3(\Delta^{1-s}) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{2\pi kr}{n}}}{1 + e^{-2\pi r}} (r^2 + \delta^2)^{1-s} dr, \quad (26)$$

$$H(s) = \int_0^\infty dy \frac{Z'(y + \delta + 1/2)}{Z(y + \delta + 1/2)} (y^2 + 2y\delta)^{1-s}, \quad (27)$$

where  $\Delta = \delta^2 + r^2$  and  $T_0 = \sum_{\{Q\}} \sum_{k=1}^{m-1} \frac{\xi^k(Q)}{2m \sin(\frac{k\pi}{m})}$ . The zero temperature contribution to effective potential is written

as

$$\begin{aligned} V_0^{(1)} &= -\frac{1}{128\pi^2} \left\{ \frac{\Lambda^2}{2} \left[ \frac{3}{2} + \ln \left( \frac{2}{\Lambda\mu^2} \right) \right] \right. \\ &\quad \left. - 2F_3(R, \phi) + \frac{4\pi R^2 H(0)}{V(\mathcal{F})} + R \left[ \frac{\Lambda}{12} + \frac{7R}{480} \right] \right. \\ &\quad \left. \left[ 1 + \ln \left( \frac{2}{\Lambda\mu^2} \right) \right] \right\} - \frac{1}{64\pi V(\mathcal{F})} \times \\ &\quad \times \sum_{\{Q\}} \sum_{k=1}^{m-1} \left\{ -f(k) R [\Lambda + Rh(k)] \left[ 1 + \ln \left( \frac{2}{\Lambda\mu^2} \right) \right] \right. \\ &\quad \left. + g(k) G_3(R, \phi) \right\} \end{aligned} \quad (28)$$

here we have used

$$\begin{aligned} f(k) &= \frac{\xi^k(Q)}{2m \sin^2(\frac{k\pi}{m})} \\ g(k) &= \frac{\xi^k(Q)}{m \sin(\frac{k\pi}{m})} \\ F_3(R, \phi) &= \int_0^\infty dr \frac{2r}{1 + e^{2\pi r}} R(Rr^2 + \Lambda) \times \\ &\quad \times \ln \left( 1 + \frac{Rr^2}{\Lambda} \right) \\ G_3(R, \phi) &= \int_{-\infty}^{+\infty} dr \frac{e^{-\frac{2\pi kr}{m}}}{1 + e^{-2\pi r}} R(Rr^2 + \Lambda) \\ &\quad \times \ln \left( 1 + \frac{Rr^2}{\Lambda} \right) \\ \Lambda &= R\delta^2 \\ h(k) &= -\frac{1}{4} + \frac{1}{2} \csc^2\left(\frac{k\pi}{m}\right). \\ R &\rightarrow -R. \end{aligned} \quad (29)$$

Now we are going to calculate the finite temperature contribution to quantum correction of the effective potential

$$\begin{aligned} V_\beta^{(1)} &= -\frac{R_H^{-4}}{16\pi^2 V(\mathcal{F}) i} \int_{R\Re = C} \zeta_R(z) \Gamma(z/2) \\ &\quad \times \Gamma\left(\frac{z}{2} - 1\right) \left(\frac{\beta}{2}\right)^{-z} R_H^z \left\{ \frac{V(\mathcal{F}) \delta^{2(2-\frac{z}{2})}}{4\pi(\frac{z}{2} - 2)} \right. \\ &\quad \left. - \frac{V(\mathcal{F})}{2\pi} \int_0^\infty r(1 - \tanh \pi r)(r^2 + \delta^2)^{1-\frac{z}{2}} dr \right. \\ &\quad \left. + \frac{(2\delta)^{\frac{3}{2}-\frac{z}{2}}}{\sqrt{\pi}\Gamma(\frac{z}{2} - 1)} \sum_{\{P\}} \sum_{k=1}^\infty \mathcal{A}(k, P) [kl(P)]^{\frac{z}{2}-\frac{3}{2}} \right. \\ &\quad \times K_{\frac{z}{2}-\frac{3}{2}}[\delta kl(P)] + \sum_{\{Q\}} \sum_{k=1}^{n-1} B(Q, k) \\ &\quad \left. \times \int_{-\infty}^{+\infty} \frac{e^{-\frac{2\pi rk}{n}}}{1 + e^{-2\pi r}} (r^2 + \delta^2)^{1-\frac{z}{2}} dr \right\} dz \end{aligned} \quad (30)$$

After the calculation of the above integrals we find that

$$V_\beta^{(1)} = V_\beta^I + V_\beta^h + V_\beta^\epsilon \quad (31)$$

where we have

$$V_\beta^I = -\frac{\pi^2 \beta^{-4}}{90} + \frac{\Lambda \beta^{-2}}{48} + \frac{R \beta^{-2}}{576} - \frac{\Lambda^{\frac{3}{2}} \beta^{-1}}{24\sqrt{2}\pi} \quad (32)$$

$$- \frac{F_1(R, \phi) \beta^{-1}}{8\pi\sqrt{2}} + \frac{1}{128\pi^2} \left\{ \frac{\Lambda^2}{2} \left[ \frac{3}{2} + \ln \left( \frac{32\pi^2}{\beta^2 \Lambda} \right) \right] \right.$$

$$+ R \left[ \frac{\Lambda}{12} + \frac{7R}{480} \right] \left[ 1 + \ln \left( \frac{32\pi^2}{\beta^2 \Lambda} \right) \right]$$

$$\left. - \Lambda^2 \gamma - 2R \gamma \left[ \frac{\Lambda}{12} + \frac{7R}{480} \right] - 2F_3(R, \phi) \right\}$$

$$+ \frac{S \Lambda^{n+1} R^{n+1} \beta^{2n-2}}{16\pi^2} + \frac{S_1 F_n(R, \phi) \beta^{2n-2}}{16\pi^2 2^{3n-1}}$$

$$V_\beta^h = \frac{R^2 T_1 I_1(R, \phi)}{32\pi V(\mathcal{F})} - \frac{R^{\frac{3}{2}} T_1 I_2(R, \phi) \beta^{-1}}{4\sqrt{2}\pi V(\mathcal{F})} \quad (33)$$

$$V_\beta^\epsilon = -\frac{1}{32\pi V(\mathcal{F})} \sum_{\{Q\}} \sum_{k=1}^{m-1} \left\{ \frac{4f(k) R \pi^2 \beta^{-2}}{3} \right. \quad (34)$$

$$\left. - 2^{\frac{3}{2}} \pi g(k) G_1(R, \phi) \beta^{-1} + \frac{f(k)}{2} [R\Lambda + R^2 h_k] \right\} \quad (35)$$

$$\times \left[ 1 - 2\gamma + \ln \left( \frac{32\pi^2}{\Lambda \beta^2} \right) - \frac{g(k)}{2} G_3(R, \phi) \right]$$

$$+ \frac{T_0 S_1 G_n(R, \phi) \beta^{2n-2}}{32\pi 2^{3n-3}}$$

where we have defined the following constants

$$F_1(R, \phi) = \int_0^\infty dr \frac{2r}{1 + e^{2\pi r}} R(Rr^2 + \Lambda)^{\frac{1}{2}} \quad (36)$$

$$G_1(R, \phi) = \int_{-\infty}^{+\infty} dr \frac{e^{-\frac{2\pi k r}{m}}}{1 + e^{-2\pi r}} R(Rr^2 + \Lambda)^{\frac{1}{2}}$$

$$I_1(R, \phi) = \int_0^\infty dy e^{-(y+\delta)kl(P)} (y^2 + 2y\delta)$$

$$I_2(R, \phi) = \int_0^\infty dy e^{-(y+\delta)kl(P)} (y^2 + 2y\delta)^{\frac{1}{2}}$$

$$F_n(R, \phi) = \int_0^\infty dr \frac{2r}{1 + e^{2\pi r}} R(Rr^2 + \Lambda)^n$$

$$G_n(R, \phi) = \int_{-\infty}^{+\infty} dr \frac{e^{-\frac{2\pi k r}{m}}}{1 + e^{-2\pi r}} R^{2n} (Rr^2 + \Lambda)^{1-n}$$

$$T_1 = \sum_{\{P\}_\Gamma} \sum_{k=1}^{\infty} \mathcal{A}(k, P)$$

$$T_0 = \sum_{\{Q\}} \sum_{k=1}^{m-1} \mathcal{B}(k, Q)$$

$$S = \sum_{n=2}^{\infty} \frac{(-1)^n \pi^{\frac{3}{2}-2n} \Gamma(n - \frac{1}{2}) \zeta(2n-1)}{(n+1)! 2^{3n-1}}$$

$$S_1 = \sum_{n=2}^{\infty} \frac{2(-1)^n \pi^{\frac{3}{2}-2n} \Gamma(n - \frac{1}{2}) \zeta(2n-1)}{(n)! 2^{3n-1}}$$

## 4 Renormalization of zero temperature effective potential

For simplicity we will consider the massless case  $m = 0$ . The renormalized zero temperature effective potential is defined by

$$V_0^R = V_c + V_0^{(1)} + V_a + \delta V \quad (37)$$

where

$$\delta V = \frac{\delta \lambda \phi^4}{4!} + \frac{\delta \xi R \phi^2}{2} + \frac{\delta a R^2}{2} \quad (38)$$

$$V_a = \frac{a R^2}{2}$$

The counterterms are fixed by the following renormalization conditions [4]

$$\delta \xi = - \left. \frac{\partial^3 V_0^{(1)}}{\partial R \partial \phi^2} \right|_{R=0, \phi=M} \quad (39)$$

$$\delta a = - \left. \frac{\partial^2 V_0^{(1)}}{\partial R^2} \right|_{R=0, \phi=M} \quad (40)$$

$$\delta \lambda = - \left. \frac{\partial^4 V_0^{(1)}}{\partial \phi^4} \right|_{R=0, \phi=M} \quad (41)$$

The counterterms were computed explicitly, they are

$$\delta \lambda = - \frac{\lambda^2}{32\pi^2} \left[ 8 + 3 \ln \left( \frac{\lambda \phi^2 \mu^2}{2} \right) \right] \quad (42)$$

$$\delta a = - \frac{1}{192} \{ 2 \lambda V_F + 6 \lambda \xi \ln(2) V_F$$

$$+ 6 \lambda \xi \ln \left( \frac{1}{\lambda M^2 \mu^2} \right) V_F - 12 \lambda \xi V_F + 6 \lambda f_k \ln(2) \pi$$

$$+ 6 \lambda f_k \ln \left( \frac{1}{\lambda M^2 \mu^2} \right) \pi$$

$$- \lambda \ln(2) V_F - \lambda \ln \left( \frac{1}{\lambda M^2 \mu^2} \right) V_F$$

$$- 12 \lambda f_k \pi \} \pi^{-2} V_F^{-1} \quad (43)$$

$$\delta \xi = - \frac{\lambda}{32\pi^2} \left[ 2 + \ln \left( \frac{\lambda M^2 \mu^2}{2} \right) \right]$$

$$\times \left[ \frac{1}{6} - \xi - \frac{f_k \pi}{V_F} \right] \quad (44)$$

where we have used the following notation

$$\begin{aligned} F_3 &\equiv \int_0^\infty dr \frac{2r}{1+e^{2\pi r}} \\ G_3 &\equiv \int_{-\infty}^{+\infty} dr \frac{e^{-\frac{2\pi k r}{m}}}{1+e^{-2\pi r}} \\ f_k &\equiv \frac{\xi^k(Q)}{2m \sin^2\left(\frac{k\pi}{m}\right)} \\ g_k &\equiv \frac{\xi^k(Q)}{m \sin\left(\frac{k\pi}{m}\right)} \\ J &= \csc^2\left(\frac{k\pi}{m}\right) \end{aligned} \quad (45)$$

Thus, the renormalized zero effective potential is a very complicated expression

$$\begin{aligned} V_0^R &= V_C + \frac{aR^2}{2} + V_0^{(1)} + \frac{\delta\lambda\phi^4}{4!} + \frac{\delta\xi R\phi^2}{2} + \frac{\delta aR^2}{2} \\ &= aR^2 + bR\phi^2 + c\phi^4 + d + (eR^2 + fR\phi^2 + g\phi^4) \\ &\quad \times \ln \left[ \frac{\lambda M^4}{\lambda\phi^2 - 2\xi R + \frac{R}{4}} \right] \end{aligned} \quad (46)$$

where  $a, b, c, d, e, f$  and  $g$  are constants.

#### 4.1 Linear curvature approximation

Taking into account only linear terms in curvature, we find the following result

$$\begin{aligned} V_0^R &\approx \frac{\lambda\phi^4}{4!} - \frac{1}{2}\xi R\phi^2 + \frac{\lambda^2\phi^4}{256\pi^2} \left[ \ln\left(\frac{\phi^2}{M^2}\right) - \frac{25}{6} \right] \\ &\quad - \frac{\lambda R\phi^2}{64\pi^2 V_F} \left[ \ln\left(\frac{\phi^2}{M^2}\right) - 3 \right] \left( \xi - \frac{1}{6} + f_k \right) \end{aligned} \quad (47)$$

the expression above agree with the result of the reference [13]. It is very ease to check that when the curvature is zero ( $R = 0$ ) we obtain the well know result of S. Coleman and E. Weinberg [24]

$$V_0^R \approx \frac{\lambda\phi^4}{4!} + \frac{\lambda^2\phi^4}{256\pi^2} \left[ \ln\left(\frac{\phi^2}{M^2}\right) - \frac{25}{6} \right] \quad (48)$$

## 5 Renormalized Finite Temperature Effective Potential

All the ambiguities of the effective potential are in the zero temperature effective potential ( $V_0$ ). Therefore we must define the finite temperature effective potential in a way that the finite temperature part is not affected by the renormalization processes, namely, when the temperature is equal zero ( $T = 0$ ) the finite effective potential goes to the zero temperature effective potential. Following the above considerations the finite temperature effective potential is defined by

$$V_{eff}^R = V_0^R + V_\beta^{(1)} \quad (49)$$

In writing the above expression becomes

$$\begin{aligned} V_{eff}^R &= aR^2 + bR\phi^2 + c\phi^4 + d + (eR^2 + fR\phi^2 \\ &\quad + g\phi^4) \ln \left[ \frac{\lambda M^4}{\lambda\phi^2 - 2\xi R + \frac{R}{4}} \right] - \frac{\pi^2\beta^{-4}}{90} + \frac{\Lambda\beta^{-2}}{48} \\ &\quad + \frac{R\beta^{-2}}{576} - \frac{\Lambda^{\frac{3}{2}}\beta^{-1}}{24\sqrt{2}\pi} - \frac{F_1(R, \phi)\beta^{-1}}{8\pi\sqrt{2}} \\ &\quad + \frac{1}{128\pi^2} \left\{ \frac{\Lambda^2}{2} \left[ \frac{3}{2} + \ln \left( \frac{32\pi^2}{\beta^2\Lambda} \right) \right] \right. \\ &\quad + R \left[ \frac{\Lambda}{12} + \frac{7R}{480} \right] \left[ 1 + \ln \left( \frac{32\pi^2}{\beta^2\Lambda} \right) \right] \\ &\quad \left. - \Lambda^2\gamma - 2R\gamma \left[ \frac{\Lambda}{12} + \frac{7R}{480} \right] - 2F_3(R, \phi) \right\} \\ &\quad + \frac{S\Lambda^{n+1}R^{n+1}\beta^{2n-2}}{16\pi^2} + \frac{S_1F_n(R, \phi)\beta^{2n-2}}{16\pi^2 2^{3n-1}} \\ &\quad + \frac{R^2T_1I_1(R, \phi)}{32\pi V(\mathcal{F})} - \frac{R^{\frac{3}{2}}T_1I_2(R, \phi)\beta^{-1}}{4\sqrt{2}\pi V(\mathcal{F})} \\ &\quad - \frac{1}{32\pi V(\mathcal{F})} \sum_{\{Q\}} \sum_{k=1}^{m-1} \left\{ \frac{4f(k)R\pi^2\beta^{-2}}{3} \right. \\ &\quad \left. - 2^{\frac{3}{2}}\pi g(k)G_1(R, \phi)\beta^{-1} + \frac{f(k)}{2}[R\Lambda + R^2h(k)] \right. \\ &\quad \left. \times [1 - 2\gamma + \ln \left( \frac{32\pi^2}{\Lambda\beta^2} \right) - \frac{g(k)}{2}G_3(R, \phi)] \right\} \\ &\quad + \frac{T_0S_1G_n(R, \phi)\beta^{2n-2}}{32\pi 2^{3n-3}} \end{aligned} \quad (50)$$

#### 5.1 High Temperature Limit

The limit of high temperature ( $\beta \rightarrow 0$ ), and when the curvature is equal zero ( $R = 0$ ) we find the result of the literature

$$\begin{aligned} V_{eff}^R &= c\phi^4 + d + g\phi^4 \ln \left[ \frac{\lambda M^4}{\lambda\phi^2} \right] - \frac{\pi^2\beta^{-4}}{90} + \frac{\Lambda\beta^{-2}}{48} \\ &\quad - \frac{\Lambda^{\frac{3}{2}}\beta^{-1}}{24\sqrt{2}\pi} + \frac{1}{128\pi^2} \left\{ \frac{\Lambda^2}{2} \left[ \frac{3}{2} + \ln \left( \frac{32\pi^2}{\beta^2\Lambda} \right) \right] - \Lambda^2\gamma \right\} \end{aligned} \quad (51)$$

less than a constant that does not depend of the temperature, the above limit result agree with the literature again [23].

## 6 The behavior of the Free Energy

The free energy  $F$  is defined in term of the effective potential as

$$F = \frac{\Omega}{h} V_{eff}^R. \quad (52)$$

We can define the order parameter ( $\langle 0|\hat{\phi}(x)|0 \rangle = \phi$ ) by the following extreme condition

$$\left. \frac{\partial F}{\partial \phi} \right|_{\phi=\phi_0} = 0 \quad (53)$$

By mean of this equation we can eliminate the order parameter by chosen the solutions to positive temperature and curvature. From above expression we find two different solutions to the order parameter, they are solutions very complicated, found using the Mathematica-5 program. When we substitute these two solutions in the free energy, we observe that the behavior of the free energies are the same to them. Furthermore, none of the roots intercept themselves. Therefore, we chose the solution of less energy.

The next step is to study the behavior of the free energy of our model. For a constant value of the temperature, the behavior of the free energy for low curvature values is shown in the graph of **Figure-1**. The behavior of the free energy

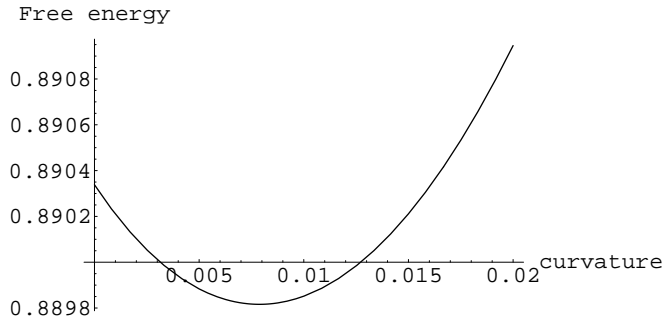


Figure 1: The behavior of the entropy with respect to the curvature for a constant value of the temperature

with respect to the temperature for a constant value of the curvature can be seen in the graph of the **Figure-2**.

## 7 Remarks and conclusions

In this paper we have calculated the one-loop finite temperature effective potential (or free energy) for a system of scalar particles on a manifold with compact hyperbolic spatial part. We have renormalized the zero temperature part of the finite temperature effective potential, and our result are according with [13]. In the limit of high temperature and a flat space-time ( $R = 0$ ) we found the well know result of the literature, see for example [23].

Carefully analyzing the free energy of our model, we realize that there exist a minimum of the free energy with respect to the curvature. We call this minimum by  $F_m(R_e, T)$ , where  $R_e$  is a stationary curvature that minimize the free energy.

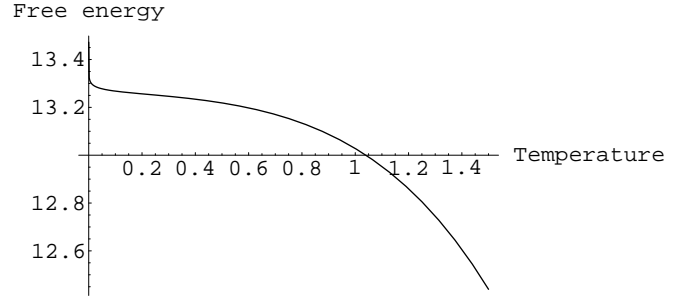


Figure 2: The behavior of the free energy with respect to the temperature for a constant value of the curvature

The equilibrium state of the system is given by a minimum of the free energy as a function of the temperature and the stationary curvature, i.e., that is the statement of the principle of minimum free energy. The stationary curvature is a function of the temperature. But, we only can estimate numerically this stationary curvature for a given value of the temperature. Because the expression is very complicate. When the temperature increases, the value of the stationary curvature increases too. Therefore, we can write the following relation of proportionality for the state of equilibrium:  $R_e \propto T$ .

This result can be once again emphasized in a very simple way. Because of the principle of minimum free energy (or maximum of the entropy in equilibrium situation), the curvature is connected with the temperature. If we consider the primordial universe with very high temperature, then the curvature will be very high too. This model is consistent with a primordial universe with large curvature and temperature, and another situation where the universe is evolving into a new situation of low temperature and low curvature. In other words, for low temperature the universe described by our model becomes flat.

## Acknowledgements

This work was supported in part by the CAPES, CNPQ and FAPESP. The authors would like to thank Andrey Bytsenko, Antônio Edson Gonçalves and B.M. Pimentel for useful discussions and comments.

## References

- [1] B.S.DeWitt. **Phys. Rep.** **19** (1975) 295.
- [2] I.L. Buchbinder and S.D. Odintsov **Class.Quantum Grav.** **2** (1985) 721-731.

- 
- [3] N.D. Birrell and P.C.W. Davies. Quantum Field in curved space. Cambridge University Press. Cambridge (1982).
  - [4] Andrey A. Bytsenko, Guido Cognola, Luciano Vanzo, Sergio Zerbini, Quantum Fields and extended objects in space-times with constant curvature spatial section. **Physics Reports** **266** (1996) 1-126.
  - [5] S.W. Hawking, Zeta function regularization of path integrals in curved spacetime, **Commun. math. Phys.** **55**, 133-148 (1977).
  - [6] A. Selberg, **J. Indian Math. Soc.** **20** (1956) 47.
  - [7] A.B. Venkov **Proc. Steklov Inst. Math.** **4**(1982) 1.
  - [8] D.A. Hejhal, The Selberg Trace formula for  $PSL(2, \mathbb{R})$  (Springer 1976), vol. I and II.
  - [9] A. Albrecht and Steinhardt P J **Phys. Rev. Lett.** **48** 1220 (1982)
  - [10] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, Effective Action in Quantum Gravity, IOP Publishing Ltd (1992)
  - [11] E.S. Abers and B.W. Lee. **Phys. Rep.** **9** (1973)
  - [12] P. Greiner Arch. Rat. Mech. and Anal. 41 (1971) 163.
  - [13] Andrei Bytsenko, Klaus Kirsten, Sergei Odintsov, **Mod. Phys. Lett. A**, vol. 8, No. 21 (1993) 2011-2021.
  - [14] J.S. Dowker and J.P. Schofield Nucl. **Physics B** **327** (1989) 267.
  - [15] G. Cognola and L. Vanzo **Mod. Phys. Lett. A** **7** (1992) 3677.
  - [16] R. Camporesi **Class. Quantum Grav.** **8** (1991) 529.
  - [17] K. Kirsten **J. Phys. A** **24** (1991) 3281.
  - [18] A.A Bytsenko, L. Vanzo and S. Zerbini. **Phys. Lett B** **291** (1992) 26.
  - [19] A.A Bytsenko, L. Vanzo and S. Zerbini. **Mod. Phys. Lett A** **7** (1992) 2669.
  - [20] G. Cognola, K. Kirsten, L. Vanzo and S. Zerbini. Finite temperature effective potential on hyperbolic space-times. **Phys. Rev D** **49** (1994) 5307.
  - [21] J.R. Ruggiero, A. Villani and A.H. Zimerman; Some comments on the application of analytic regularization to the Casimir forces; **J. Phys. A: Math. Gen** **13** (1980) 761-766.
  - [22] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini; Zeta Regularization Techniques with Application; World Scientific Publishing (1994).
  - [23] P. Ramond; Field Theory: A Modern Primer, second edition, vol. 74 Addison-Wesley Publishing Company (1990).
  - [24] S. Coleman and E. Weinberg, **Phys. Rev. D** **7** 1888 (1973).